A DUALITY OF THE TWISTED GROUP ALGEBRA OF THE SYMMETRIC GROUP AND A LIE SUPERALGEBRA

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0. Introduction

The "character values" of the irreducible projective representations of $\mathfrak{S}_k$, the symmetric group of degree k, were determined by I. Schur using Schur's Q-functions, which are indexed by the distinct partitions of k, [10], in a way analogous to Frobenius' formula for the character values of the ordinary irreducible representations of $\mathfrak{S}_k$ [2]. Behind Frobenius' formula exists a duality relation of $\mathfrak{S}_k$ and the general linear group $GL(n)$ (the Schur-Weyl duality). It is natural to expect the existence of an analogous duality relation between the twisted group algebra $\mathcal{A}_k$ (cf. (1.2)) of $\mathfrak{S}_k$ and some algebra, behind Schur's method. A. N. Sergeev showed that a twisted group algebra $\mathcal{B}_k$ (cf. (1.3)) of the hyperoctahedral group H_k and a Lie superalgebra $\mathfrak{q}(n)$ (cf. §1, **G**) act on the k-th tensor product $W = V^{\otimes k}$ of the $2n$-dimensional natural representation $V = \mathbb{C}^n \oplus \mathbb{C}^n$ of $\mathfrak{q}(n)$, as mutual commutants of each other [11] (in the sense of $\mathbb{Z}/2\mathbb{Z}$-graded algebras, see §1, **E**). This result motivated our work.

In this paper, we establish a duality relation between $\mathcal{A}_k$ and $\mathfrak{q}(n)$ on a subspace of W, and give a representation-theoretic explanation of Schur's identity (1.6) adapted to the context of $\mathbb{Z}/2\mathbb{Z}$-graded representations by T. Józefiak (Corollary 4.2).

In §3, we construct an isomorphism $\mathcal{B}_k \cong \mathcal{C}_k \dot{\otimes} \mathcal{A}_k$ of $\mathbb{Z}/2\mathbb{Z}$-graded algebras (Theorem 3.2), where $\mathcal{C}_k$ is the 2^k-dimensional Clifford algebra and $\dot{\otimes}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product (cf. §1, **E**). This isomorphism does imply an embedding $\mathcal{A}_k \hookrightarrow \mathcal{B}_k$, although $\mathcal{A}_k$ does not sit in $\mathcal{B}_k$ in an obvious manner (cf. §1, **D**). Then, we give a simple relation between the $\mathbb{Z}/2\mathbb{Z}$-graded irreducible representations of $\mathcal{B}_k$ and $\mathcal{A}_k$ (Proposition 3.5). Note that J. R. Stembridge constructed the non-graded simple modules of the underlying algebra $|\mathcal{B}_k|$ as submodules of non-graded tensor products of modules of three twisted group algebras of H_k [13],

but his description of the simple $|\mathcal{B}_k|$-modules does not immediately show a simple relation between $\mathcal{A}_k$ and $\mathcal{B}_k$.

In §4, we give a submodule W' of W, as a simultaneous eigenspace of $2^{\lfloor k/2\rfloor}$ involutions contained in $\mathcal{C}_k$, where $\lfloor k/2\rfloor$ denotes the largest integer not exceeding $k/2$, and show that $\mathcal{A}_k$ and $\mathfrak{q}(n)$ act on W' as mutual commutants of each other in the context of $\mathbb{Z}/2\mathbb{Z}$-graded algebras (Theorem 4.1).

In this paper, all vector spaces, and associative algebras, and representations in this paper are assumed to be finite dimensional unless otherwise stated, and over the complex number field $\mathbb{C}$.

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1. Preliminaries and notations

A. The hyperoctahedral group H_k. The hyperoctahedral group H_k is the group generated by t and the s_i, $1 \le i \le k-1$, subject to relations

$$\begin{aligned} &t^2 = s_i^2 = 1 \quad (1 \le i \le k-1), \\ &(s_i s_{i+1})^3 = 1 \quad (1 \le i \le k-2), \quad (s_i s_j)^2 = 1 \quad (|i-j| \ge 2), \\ &(ts_i)^2 = 1 \quad (2 \le i \le k-1), \quad (ts_1)^4 = 1. \end{aligned} \tag{1.1}$$

H_k is isomorphic to the Weyl group of type B_k or C_k, and is also sometimes called the group of signed permutations. The subgroup of H_k generated by the s_i, $1 \le i \le k-1$, is isomorphic to the symmetric group of degree k, which we denote by $\mathfrak{S}_k$.

B. Partitions. Let P_k denote the set of all partitions of k (see [8, p. 1]), and put $P = \coprod_{k\ge 0} P_k$. For $\lambda \in P$, we write $l(\lambda)$ for the length of λ, namely the number of nonzero parts of λ. Also we write $|\lambda| = k$ if $\lambda \in P_k$. Let DP_k and OP_k denote the distinct partitions (or strict partitions, namely partitions whose parts are distinct) and the odd partitions (namely partitions whose parts are all odd) of k respectively. Let DP_k^+ and DP_k^- be the sets of all $\lambda \in DP_k$ such that $(-1)^{k-l(\lambda)} = +1$ and -1 respectively. Note that $(-1)^{k-l(\lambda)}$ equals the signature of permutations with cycle type λ. We also put $DP = \coprod_{k\ge 0} DP_k$ and $OP = \coprod_{k\ge 0} OP_k$. Note that these notations, DP and OP, were used by Stembridge in [12], [13].

C. The subring Ω of the ring of the symmetric functions. Let Λ denote the ring of the symmetric functions with coefficients in $\mathbb{C}$. Note that our Λ is the scalar extension of the Λ in [8], which is a $\mathbb{Z}$-algebra, to $\mathbb{C}$. We have $\Lambda = \bigoplus_{k\ge 0} \Lambda^k$, where Λ^k denotes the subspace of Λ consisting of the homogeneous elements of degree k.

For each $r \ge 1$, let p_r denote the r-th power sum $\sum_{i\ge 1} x_i^r \in \Lambda$. The p_r, $r \ge 1$, are algebraically independent over $\mathbb{C}$ and we have $\Lambda = \mathbb{C}[p_1, p_2, \dots]$. For each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in P$, let $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$. Then $\{p_\lambda\, ;\, \lambda \in P\}$ is a basis of Λ (cf. [8, §I, sect. 2.]).

Let Ω denote the subring of Λ generated by the power sums of odd degrees, namely the p_r, $r = 1, 3, 5, \ldots$. Put $\Omega^k = \Omega \cap \Lambda^k$. Then we have $\Omega = \bigoplus_{k \geq 0} \Omega^k$, and $\{p_\lambda \mid \lambda \in OP\}$ is a basis of Ω.

For $\lambda \in DP$, let $Q_\lambda \in \Lambda$ denote Schur's Q-function indexed by λ (cf. [10], [12, sect. 6]). Then $\{Q_\lambda \mid \lambda \in DP\}$ is also a basis of Ω.

D. Twisted group algebras $\mathcal{A}_k$ and $\mathcal{B}_k$. For each $k \geq 1$, let $\mathcal{A}_k$ denote the associative algebra generated by the elements γ_i, $1 \leq i \leq k-1$, subject to relations

$$\begin{aligned} &\gamma_i^2 = -1 \ \ (1 \leq i \leq k-1), \ \ (\gamma_i \gamma_{i+1})^3 = -1 \ \ (1 \leq i \leq k-2), \\ &(\gamma_i \gamma_j)^2 = -1 \ \ (|i-j| \geq 2). \end{aligned} \tag{1.2}$$

$\mathcal{A}_k$ is isomorphic to a twisted group algebra of $\mathfrak{S}_k$ (see below). We regard $\mathcal{A}_k$ as a $\mathbb{Z}_2$-graded algebra by giving degree $1 \in \mathbb{Z}_2$ to the generators γ_i for all $1 \leq i \leq k-1$. In the following, we abbreviate $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}_2$.

For each $k \geq 1$, let $\mathcal{B}_k$ denote the associative algebra generated by τ and the σ_i, $1 \leq i \leq k-1$, subject to relations

$$\begin{aligned} &\tau^2 = \sigma_i^2 = 1 \quad (1 \leq i \leq k-1), \quad (\sigma_i \sigma_{i+1})^3 = 1 \quad (1 \leq i \leq k-2), \\ &(\sigma_i \sigma_j)^2 = 1 \quad (|i-j| \geq 2), \quad (\tau \sigma_i)^2 = 1 \quad (2 \leq i \leq k-1), \\ &(\tau \sigma_1)^4 = -1. \end{aligned} \tag{1.3}$$

$\mathcal{B}_k$ is isomorphic to a twisted group algebra of H_k (again see below). We regard $\mathcal{B}_k$ as a $\mathbb{Z}_2$-graded algebra by giving degree 1 to the generator τ and degree 0 to the generator σ_i for all $1 \leq i \leq k-1$.

Remark. The structures of $H^2(\mathfrak{S}_k, \mathbb{C}^\times)$ and $H^2(H_k, \mathbb{C}^\times)$ were clarified by I. Schur [10] and J. W. Davies, A. O. Morris [1] respectively, as follows:

$$H^2(\mathfrak{S}_k, \mathbb{C}^\times) \cong \begin{cases} 0 & \text{for } \ k \leq 3 \\ \mathbb{Z}/2\mathbb{Z} & \text{for } \ k \geq 4 \end{cases} \tag{1.4}$$

$$H^2(H_k, \mathbb{C}^\times) \cong \begin{cases} 0 & \text{if } k = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k = 2 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } k = 3 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } k \geq 4. \end{cases} \tag{1.5}$$

If $k \geq 4$, then any twisted group algebra $\mathbb{C}^\alpha \mathfrak{S}_k$ of $\mathfrak{S}_k$ with a non-trivial 2-cocycle α is isomorphic to $\mathcal{A}_k$ (cf. [10], [12, Lem. 1.1]). If $k \leq 3$, then $\mathcal{A}_k$ is isomorphic to the ordinary group algebra $\mathbb{C}\mathfrak{S}_k$, via $\gamma_i \mapsto \sqrt{-1} s_i$.

Moreover, if $k \geq 2$, then $\mathcal{B}_k$ is isomorphic to a twisted group algebra $\mathbb{C}^\alpha H_k$ of H_k with a non-trivial 2-cocycle α (cf. [13, Prop. 1.1]). Note that the cocycle associated with $\mathcal{B}_k$ does not restrict to the cocycle associated with $\mathcal{A}_k$.

In the following, we consistently use the formulation of the $\mathbb{Z}_2$-graded representations, namely the $\mathbb{Z}_2$-graded modules of $\mathbb{Z}_2$-graded algebras (superalgebras), as was used in [3] and [4]. This formulation is slightly different from the traditional parametrization of the non-graded representations, but there exists a explicit relation between the $\mathbb{Z}_2$-graded representations of a $\mathbb{Z}_2$-graded algebra A and the non-graded representations of the underlying algebra $|A|$ (cf. [3, Lem. (2.8), Cor. (2.16), Prop. (2.17)], [4, Prop. 2.5, Cor. 2.6]).

E. Semisimple superalgebras. This theory was developed by T. Józefiak in [3], which we mostly follow. A $\mathbb{Z}_2$-graded algebra A, which is called a **superalgebra** in this paper, is called **simple** if it has no $\mathbb{Z}_2$-graded two-sided ideals except itself and 0.

Let V be a $\mathbb{Z}_2$-graded vector space, namely a vector space with a fixed direct sum decomposition $V = V_0 \oplus V_1$. We write $\mathbf{dim}\, V$ for the pair $(\dim V_0, \dim V_1)$. If $W = W_0 \oplus W_1$ is another $\mathbb{Z}_2$-graded vector space, then the vector space $\mathrm{Hom}(V, W)$ consisting of all linear maps from V to W has a $\mathbb{Z}_2$-gradation defined as follows:

$$\mathrm{Hom}^\alpha(V, W) = \{f \in \mathrm{Hom}(V, W)\,;\, f(V_\beta) \subset W_{\alpha+\beta} \text{ for all } \beta \in \mathbb{Z}_2\}$$

for each $\alpha \in \mathbb{Z}_2$. In particular, it can be easily checked that the endomorphism algebra $\mathrm{End}(V) = \mathrm{Hom}(V, V)$, which is isomorphic to the full matrix ring M_{n+m} where $\mathbf{dim}\, V = (n, m)$, can be regarded as a superalgebra with this gradation. This superalgebra is denoted by $M(n, m)$. It is clear that $M(n, m)$ is simple, since the underlying algebra $|M(n, m)| = M_{n+m}$ is a simple algebra.

There exist another type of simple superalgebras. Let $Q(n)$ denote a subsuperalgebra of $M(n, n)$ which consists of all $2n \times 2n$-matrices of the form $\begin{pmatrix} C & D \\ D & C \end{pmatrix}$, where C and D are $n \times n$-matrices. It is easy to show that $Q(n)$ is a simple superalgebra.

Theorem 1.1. (cf. [3, Th. 2.6], [4, Th. 2.1], [15]) *A simple superalgebra is isomorphic to either $M(n, m)$ for some n, m or $Q(n)$ for some n.*

A simple superalgebra A is said to be of **type** M (resp. of **type** Q) if A is isomorphic to $M(n, m)$ for some n, m (resp. is isomorphic to $Q(n)$ for some n).

Let V be an A-**module**, namely a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ together with an algebra homomorphism $\rho\colon A \to \mathrm{End}(V)$, which preserves $\mathbb{Z}_2$-gradations. Then we call ρ a **representation** of A in V, and simply write $\rho(a)v = av$ for all $a \in A$ and $v \in V$. A $\mathbb{Z}_2$-graded subspace W of V is called an A-**submodule** if it is stable under $\rho(A)$. We say that V is **simple** if it has no A-submodules except itself and 0.

Let V and W be two A-modules. For each $\alpha \in \mathbb{Z}_2$, let $\mathrm{Hom}^\alpha_A(V, W)$ denote the subspace of $\mathrm{Hom}^\alpha(V, W)$ consisting of all elements $f \in \mathrm{Hom}^\alpha(V, W)$ such that $f(av) = (-1)^{\alpha\cdot\beta} a f(v)$ for all $a \in A_\beta$ ($\beta \in \mathbb{Z}_2$), $v \in V$. Put $\mathrm{Hom}^{\cdot}_A(V, W) = \mathrm{Hom}^0_A(V, W) \oplus \mathrm{Hom}^1_A(V, W)$ and put $\mathrm{End}^{\cdot}_A(V) = \mathrm{Hom}^{\cdot}_A(V, V)$. It is clear that $\mathrm{End}^{\cdot}_A(V)$ is a subsuperalgebra of $\mathrm{End}(V)$ with this gradation. We call $\mathrm{End}^{\cdot}_A(V)$ the **supercentralizer** of $\rho(A)$ or A in $\mathrm{End}(V)$, where ρ is the representation of A

associated with V. The **shift** of V, denoted by $\overline{V}$ in this paper, is defined to be the same vector space as V with the switched grading, namely $\overline{V}_0 = V_1$ and $\overline{V}_1 = V_0$, and with the homomorphism $\overline{\rho}\colon A \to \mathrm{End}(\overline{V})$ defined by $\overline{\rho}(a) = (-1)^\alpha \rho(a)$ for $a \in A_\alpha$, where ρ is the superalgebra homomorphism associated with V. Note that we have $\mathrm{Hom}^0_A(\overline{V}, W) = \mathrm{Hom}^1_A(V, W)$ and $\mathrm{Hom}^1_A(\overline{V}, W) = \mathrm{Hom}^0_A(V, W)$. Two A-modules V and W are called **isomorphic** (resp. **strictly isomorphic**) if there exists an invertible linear map $f \in \mathrm{Hom}^{\cdot}_A(V, W)$ (resp. $f \in \mathrm{Hom}^0_A(V, W)$). If this is the case, we write $V \cong_A W$ (resp. $V \stackrel{\cdot}{\cong}_A W$). In view of the following theorem, if V and W are simple A-modules, we have $V \cong_A W$ if and only if $V \stackrel{\cdot}{\cong}_A W$ or $\overline{V} \stackrel{\cdot}{\cong}_A W$.

The following theorem contains an analogue of Schur's Lemma.

Theorem 1.2. (cf. [3, Prop. 2.17], [4, Prop. 2.5, Cor. 2.6]) *Let $V = V_0 \oplus V_1$ be a simple A-module.*

(1) *The supercentralizer $\mathrm{End}^{\cdot}_A(V)$ is isomorphic to $M(1,0) \cong \mathbb{C}$, if and only if V_0 is not isomorphic to V_1 as an A_0-module.*

(2) *The supercentralizer $\mathrm{End}^{\cdot}_A(V)$ is isomorphic to $Q(1) \cong \mathcal{C}_1$, if and only if V_0 is isomorphic to V_1 as an A_0-module.*

If a simple A-module V satisfies either of the equivalent conditions in (1) (resp. (2)), then we say that it is of **type** M (resp. of **type** Q). If V is a simple A-module of type M, then so is $\overline{V}$ and we have $V \stackrel{\cdot}{\ncong}_A \overline{V}$. If V is a simple A-module of type Q, then so is $\overline{V}$ and we have $V \stackrel{\cdot}{\cong}_A \overline{V}$. In both cases we have $V \cong_A \overline{V}$.

Let A and B be two superalgebras and let V (resp. W) be an A (resp. B)-module. The vector space $A \otimes B$ can be turned into a superalgebra, where the grading is defined by $(A \otimes B)_\alpha = \oplus_{\beta+\gamma=\alpha} A_\beta \otimes B_\gamma$ for each $\alpha \in \mathbb{Z}_2$, and the multiplication is defined by

$$(a \otimes b)(c \otimes d) = (-1)^{\beta\cdot\gamma} ac \otimes bd$$

for any $a \in A$, $b \in B_\beta$, $c \in A_\gamma$, and $d \in B$ (β, $\gamma \in \mathbb{Z}_2$). This superalgebra is called the **supertensor product** of A and B and denoted by $A \dot{\otimes} B$. Note that we have an isomorphism of superalgebras $\omega_{A,B}\colon A \dot{\otimes} B \to B \dot{\otimes} A$ determined by $\omega_{A,B}(a \dot{\otimes} b) = (-1)^{\alpha\cdot\beta} b \dot{\otimes} a$ for all homogeneous elements $a \in A_\alpha$ and $b \in B_\beta$ (α, $\beta \in \mathbb{Z}_2$). The following states that the supertensor product of simple superalgebras is again a simple superalgebra.

Theorem 1.3. [3, Prop. 2.10], [15] *There exist isomorphisms of superalgebras*

(a) $M(r,s) \dot{\otimes} M(p,q) \cong M(rp + sq, rq + sp)$,
(b) $M(r,s) \dot{\otimes} Q(n) \cong Q(rn + sn)$,
(c) $Q(m) \dot{\otimes} Q(n) \cong M(mn, mn)$.

Moreover, $V \otimes W$ can be regarded as a $A \dot{\otimes} B$-module as follows:

$$(a \dot{\otimes} b)(v \otimes w) = (-1)^{\beta\cdot\alpha} av \otimes bw$$

for any $a \in A$, $b \in B_\beta$, $v \in V_\alpha$, and $w \in W$ (α, $\beta \in \mathbb{Z}_2$), and $(V \otimes W)_\alpha = \oplus_{\beta+\gamma=\alpha} V_\beta \otimes W_\gamma$ for $\alpha \in \mathbb{Z}_2$. This $A \dot{\otimes} B$-module is called the **supertensor product** of V and W and denoted by $V \dot{\otimes} W$. Note that the supertensor product is symmetric, namely the $B \dot{\otimes} A$-module obtained from the $A \dot{\otimes} B$-module $V \dot{\otimes} W$ via $\omega_{B,A}$ is isomorphic to $W \dot{\otimes} V$ by the map determined by $v \dot{\otimes} w \mapsto (-1)^{\alpha\cdot\beta} w \dot{\otimes} v$ for all homogeneous $v \in V_\alpha$ and $w \in W_\beta$ (α, $\beta \in \mathbb{Z}_2$).

The following follows from Schur's lemma.

Theorem 1.4. *Let A, B be superalgebras, and put $C = A \dot{\otimes} B$. Let U (resp. W) be a simple A-module (resp. simple B-module), and put $V = U \dot{\otimes} W$.*

- (a) *If U, W are of type M, then V is a simple C-module of type M. We have $\overline{U} \dot{\otimes} W \dot{\cong}_C U \dot{\otimes} \overline{W} \dot{\cong}_C \overline{V}$, $\overline{U} \dot{\otimes} \overline{W} \dot{\cong}_C V$ and $V|_A \dot{\cong}_A U^{\oplus k'} \oplus \overline{U}^{\oplus l'}$, $V|_B \dot{\cong}_B W^{\oplus k} \oplus \overline{W}^{\oplus l}$ where $(k,l) = \mathbf{dim}\, U$, $(k',l') = \mathbf{dim}\, W$.*
- (b) *If one of U and W is of type M and the other is of type Q, then from the symmetry of the supertensor product we have only to state for the case where U is of type M and W is of type Q. Then V is a simple C-module of type Q. We have $\overline{U} \dot{\otimes} W \dot{\cong}_C V$ and $V|_A \dot{\cong}_A U^{\oplus n'} \oplus \overline{U}^{\oplus n'}$, $V|_B \dot{\cong}_B W^{\oplus k+l}$ where $(k,l) = \mathbf{dim}\, U$, $(n',n') = \mathbf{dim}\, W$.*
- (c) *Suppose U and W are of type Q. Fix $x \in \mathrm{End}^1_A(U)$ (resp. $y \in \mathrm{End}^1_B(W)$) satisfying $x^2 = -1$ (resp. $y^2 = -1$). Then $x \dot{\otimes} y \in \mathrm{End}^0_C(U \dot{\otimes} W)$ satisfies $(x \dot{\otimes} y)^2 = -1$. Let $V^\pm$ be the $(\pm\sqrt{-1})$-eigenspace of $x \dot{\otimes} y$ respectively. Then $V = V^+ \oplus V^-$ is a decomposition into simple C-modules, both of type M. We have $\overline{V^+} \dot{\cong}_C V^-$ and $V^+|_A \dot{\cong}_A V^-|_A \dot{\cong}_A U^{\oplus n'}$, $V^+|_B \dot{\cong}_B V^-|_B \dot{\cong}_B U^{\oplus n}$ where $(n,n) = \mathbf{dim}\, U$, $(n',n') = \mathbf{dim}\, W$.*

Moreover, the above construction gives all simple $A \dot{\otimes} B$-modules.

Let $\mathrm{Irr}\, A$ denotes the set of all isomorphism classes (not strict isomorphism classes) of simple A-modules for any superalgebra A.

Corollary 1.5. *We have a bijection* $\mathrm{Irr}\, A \dot{\otimes} B \xrightarrow{\sim} \mathrm{Irr}\, A \times \mathrm{Irr}\, B$.

For any A-module V, the following conditions are equivalent.

1. V is a sum of simple A-submodules,
2. V is a direct sum of simple A-submodules,
3. For any A-submodule W of V, there exists a A-submodule W' of V such that $V = W \oplus W'$.

We call an A-module V **semisimple** if it satisfies one of the above equivalent conditions (1)-(3).

For any superalgebra A, the following conditions are equivalent (cf. [3, Prop. 2.4], [3, Cor. 2.12], [4, Th. 2.2, Cor. 2.3]):

1. A is a semisimple (regular) A-module, namely A is a direct sum of simple A-submodules,
2. every A-module is semisimple,
3. A is a direct sum of simple superalgebras.

We call a superalgebra A **semisimple** if it satisfies one of the above equivalent conditions (1)-(3). Note that the supertensor product of two semisimple superalgebras is semisimple.

Theorem 1.6. (cf. [3, Cor. 2.12], [4, Th. 2.2, Cor. 2.3]) *Let A be a semisimple superalgebra.*

(1) *There exist integers $m, q \geq 0$ and $k_i, l_i \geq 0$ $(1 \leq i \leq m)$ and $n_j \geq 0$ $(1 \leq j \leq q)$ such that*

$$A \cong \bigoplus_{i=1}^{m} M(k_i, l_i) \oplus \bigoplus_{j=1}^{q} Q(n_j)$$

as a superalgebra. The following data are determined by A, and conversely the following data determine A: (a) $m = m(A)$, (b) $q = q(A)$, (c) *the multiset of the unordered pairs $\{k_i, l_i\}$, and* (d) *the multiset of the numbers n_j.*

(2) *The number of the isomorphism classes of the simple A-modules is equal to $m(A) + q(A)$.*

Note that the set of simple direct summands of A, isomorphic to $M(k_i, l_i)$ $(1 \leq i \leq m)$ and $Q(n_j)$ $(1 \leq j \leq q)$, is uniquely determined by A. We call them the **simple components** of A. We should also remark the equivalence of the semisimplicity of a superalgebra to that of the underlying algebra:

Proposition 1.7. (cf. [3, Cor. 2.16]) *A superalgebra A is semisimple if and only if an underlying algebra $|A|$ is semisimple.*

Let A be a semisimple superalgebra, and let V be an A-module. Then there exist a finite number of simple A-submodules $U_1, U_2, \dots, U_l$ of V such that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_l$ as an A-module. For each $1 \leq r \leq l$, put $U(V)_r = \sum_{U_s \cong_A U_r} V_s$. Then we can choose a subset R of $\{1, 2, \dots, l\}$ such that $V = \bigoplus_{r \in R} U(V)_r$. Note that the set of A-submodules $U(V)_r$, $r \in R$, is uniquely determined by V. The $U(V)_r$ are called the A-**homogeneous components** of V.

In the next section, §2, we will precisely state the double centralizer theorem for semisimple superalgebras, in which we will show that there exists a $1-1$ correspondence between the simple A-modules and the simple $\mathrm{End}^{\cdot}_A(V)$-modules appearing in an A-module V.

F. Character formulas for $\mathcal{A}_k$ and $\mathcal{B}_k$. Let A be a superalgebra: $A = A_0 \oplus A_1$, and let V be an A-module. Define a linear map $\mathrm{Ch}[V]\colon A \to \mathbb{C}$ by $\mathrm{Ch}[V](a) = \mathrm{tr}_V(a)$ for all $a \in A$. It is called the **character** of the A-module V. $\mathrm{Ch}[V]$ only depends on the isomorphism class of V. We also say that V **affords** the character $\mathrm{Ch}[V]$. Let $\{U_r\}_{r \in R}$ be a complete set of representatives of the isomorphism classes of the simple A-modules. Let $Z_0(A^*)$ denote the subspace of $A^* = \mathrm{Hom}(A, \mathbb{C})$ consisting of all elements $f\colon A \to \mathbb{C}$ such that $f(ab) = f(ba)$ for all a, $b \in A$ and such that $f|_{A_1} = 0$. The character of any A-module belongs to $Z_0(A^*)$. If A is semisimple, then $\{\mathrm{Ch}[U_r]\,;\, r \in R\}$ is a basis of $Z_0(A^*)$. In this case, the isomorphism classes of A-modules are uniquely determined by their characters.

Let $\mathfrak{S}'_k$ be the subgroup of $(\mathcal{A}_k)^\times$ generated by $-1, \gamma_1, \dots, \gamma_{k-1}$ (-1 is not necessary if $k \geq 4$). Then $\mathfrak{S}'_k$ is a double cover (a central extension with a $\mathbb{Z}_2$

kernel) of $\mathfrak{S}_k$ with a group homomorphism $\pi\colon \mathfrak{S}'_k \to \mathfrak{S}_k$ defined by $\pi(-1) = 1$ and $\pi(\gamma_j) = s_j$ for all $1 \leq j \leq k-1$. If $k \geq 4$, then $\mathfrak{S}'_k$ is a representation group of $\mathfrak{S}_k$. Let γ^μ be an element of $\mathfrak{S}'_k$ defined by taking a particular reduced expression of a permutation of cycle type μ and replacing each s_i with γ_i, namely:

$$\gamma^\mu = (\gamma_1 \cdots \gamma_{\mu_1-1})(\gamma_{\mu_1+1} \cdots \gamma_{\mu_1+\mu_2-1}) \cdots (\gamma_{\mu_1+\cdots+\mu_{r-1}+1} \cdots \gamma_{\mu_1+\cdots+\mu_r-1})$$

where $r = l(\mu)$. Let φ be a character of an $\mathcal{A}_k$-module. Then we have $\varphi(\gamma^\mu) = 0$ unless $\mu \in OP_k$ (cf. [10], [12]). I. Schur explicitly described the characters of the simple $\mathcal{A}_k$-modules. We review Schur's result in the form translated by T. Józefiak into the language of $\mathbb{Z}_2$-graded representations [4]. Let $\varepsilon\colon DP_k \to \mathbb{Z}_2$ be defined by $\varepsilon(\nu) = 0$ if $\nu \in DP_k^+$ and $\varepsilon(\nu) = 1$ if $\nu \in DP_k^-$.

Theorem 1.8. [4], [10] *Define the $\varphi_\nu \in Z_0((\mathcal{A}_k)^*)$, $\nu \in DP_k$, by*

$$(\sqrt{2})^{l(\mu)} p_\mu = \sum_{\nu \in DP_k} \varphi_\nu(\gamma^\mu)(\sqrt{2})^{-l(\nu)-\varepsilon(\nu)} Q_\nu \tag{1.6}$$

for all $\mu \in OP_k$, where Q_ν denotes the Schur Q-function (cf. [10], [12]*). Then the φ_ν, $\nu \in DP_k$, give all irreducible characters of $\mathcal{A}_k$.*

For each $\nu \in DP_k$, fix a simple $\mathcal{A}_k$-module V_ν which affords the irreducible character φ_ν of $\mathcal{A}_k$. Then V_ν is of type M (resp. type Q) if $\nu \in DP_k^+$ (resp. $\nu \in DP_k^-$).

Let H'_k be the subgroup of $(\mathcal{B}_k)^\times$ generated by -1, τ, $\sigma_1, \ldots, \sigma_{k-1}$ (-1 is not necessary if $k \geq 2$). Then H'_k is a double cover of H_k with a group homomorphism $\pi\colon H'_k \to H_k$ defined by $\pi(-1) = 1$, $\pi(\tau) = t$, and $\pi(\sigma_j) = s_j$ for all $1 \leq j \leq k-1$. Let $\sigma^{(\lambda,\mu)}$ denote a similarly defined element of H'_k mapping to a representative of the conjugacy class of H_k indexed by (λ, μ):

$$\begin{aligned}
&\sigma^{(\lambda,\mu)} = x_1 x_2 \cdots y_1 y_2 \cdots\ ,\\
&x_i = \sigma_{\lambda_1+\cdots+\lambda_{i-1}+1} \cdots \sigma_{\lambda_1+\cdots+\lambda_i-1},\\
&y_i = \sigma_{|\lambda|+\mu_1+\cdots+\mu_{i-1}+1} \cdots \sigma_{|\lambda|+\mu_1+\cdots+\mu_i-1} \tau_{|\lambda|+\mu_1+\cdots+\mu_i},\\
&\tau_j = \sigma_{j-1} \cdots \sigma_1 \tau \sigma_1 \cdots \sigma_{j-1}\ \ .
\end{aligned}$$

Let ψ be a character of a $\mathcal{B}_k$-module. Then we have $\psi(\sigma^{(\lambda,\mu)}) = 0$ unless $\lambda \in OP_k$ and $\mu = \phi$ (cf. [1], [13]). A. N. Sergeev showed the following explicit formula for the characters of the simple $\mathcal{B}_k$-modules using his duality relation which we will review in the next subsection, §1, **G**. Define a map $d\colon P_k \to \mathbb{Z}_2$ by $d(\nu) = 1$ if $l(\nu)$ is odd, and $d(\nu) = 0$ if $l(\nu)$ is even.

Theorem 1.9. *Define the functions $\psi_\nu \in Z_0((\mathcal{B}_k)^*)$, $\nu \in DP_k$, by*

$$2^{l(\mu)} p_\mu = \sum_{\nu \in DP_k} \psi_\nu(\sigma^{(\mu,\phi)})(\sqrt{2})^{-l(\nu)-d(\nu)} Q_\nu \tag{1.7}$$

for all $\mu \in OP_k$. Then the ψ_ν, $\nu \in DP_k$, give all irreducible characters of $\mathcal{B}_k$.

For each $\nu \in DP_k$, fix a simple $\mathcal{B}_k$-module W_ν which affords the character ψ_ν. Then W_ν is of type M (resp. type Q) if $l(\nu)$ is even (resp. odd).

G. Sergeev's duality relation. First we introduce the Lie superalgebra $\mathfrak{q}(n)$, sometimes called the queer Lie superalgebra. A standard reference for Lie superalgebras is [7].

Let $\mathfrak{gl}(n/m)$ (denoted by $l(n,m)$ in [7]) denote the Lie superalgebra whose underlying vector space is that of (the superalgebra) $M(n,m)$ and the Jacobi product $[\ ,\ ]\colon \mathfrak{gl}(n/m)\times\mathfrak{gl}(n/m)\to\mathfrak{gl}(n/m)$ is defined by $[X,Y]=XY-(-1)^{\alpha\cdot\beta}YX$ for all $X\in\mathfrak{gl}(n/m)_\alpha$ and $Y\in\mathfrak{gl}(n/m)_\beta$ $(\alpha,\beta\in\mathbb{Z}_2)$. Let $\mathfrak{q}(n)$ denote the Lie superalgebra for the superalgebra $Q(n)$ in the same manner. Let $\mathcal{U}_n=\mathcal{U}(\mathfrak{q}(n))$ denote the universal enveloping algebra of $\mathfrak{q}(n)$, which can be regarded as a superalgebra (cf. [7]).

The Lie superalgebra $\mathfrak{q}(n)$ naturally acts on a $2n$-dimensional space V with a fixed $\mathbb{Z}_2$-gradation $V=V_0\oplus V_1$, $\dim V_0=\dim V_1=n$. Therefore $\mathcal{U}_n$ also acts on V. Let W denote the k-fold supertensor product $W=V^{\dot\otimes k}$ of V. We define a representation $\Theta\colon\mathcal{U}_n\to\mathrm{End}(W)$ by

$$\Theta(X)(v_1\dot\otimes\cdots\dot\otimes v_k)=\sum_{j=1}^{k}(-1)^{\alpha\cdot(\beta_1+\cdots+\beta_{j-1})}v_1\dot\otimes\cdots\dot\otimes\overset{j}{\overbrace{Xv_j}}\dot\otimes\cdots\dot\otimes v_k \tag{1.8}$$

for all $X\in\mathfrak{q}(n)_\alpha$ and $v_i\in V_{\beta_i}$ $(1\le i\le k)$ and α, $\beta_i\in\mathbb{Z}_2$ $(1\le i\le k)$. Actually this action can be defined using a superalgebra homomorphism $\Delta\colon\mathcal{U}_n\to\mathcal{U}_n\dot\otimes\mathcal{U}_n$ called the coproduct, determined by $\Delta(X)=1\dot\otimes X+X\dot\otimes 1$ for all $X\in\mathfrak{q}(n)$. Namely, if we put $\Delta^{(k)}=(\overbrace{\mathrm{id}\dot\otimes\cdots\dot\otimes\mathrm{id}}^{k-2}\dot\otimes\Delta)\circ\cdots\circ(\mathrm{id}\dot\otimes\Delta)\circ\Delta\colon\mathcal{U}_n\to\mathcal{U}_n^{\dot\otimes k}$, then we have $\Theta(X)(v_1\dot\otimes\cdots\dot\otimes v_k)=\Delta^{(k)}(X)(v_1\dot\otimes\cdots\dot\otimes v_k)$. Note that $\mathcal{U}_n$ is an infinite dimensional superalgebra. However, for a fixed number k, $\mathcal{U}_n$ acts on W through its finite dimensional image in $\mathrm{End}(W)$. Therefore we can use the results in §1, **E** on finite dimensional superalgebras and their finite dimensional modules.

Next define a representation $\Psi\colon\mathcal{B}_k\to\mathrm{End}(W)$ by

(1.9)
$$\begin{aligned}
&\Psi(\tau)(v_1\dot\otimes\cdots\dot\otimes v_k)=(Pv_1)\dot\otimes v_2\dot\otimes\cdots\dot\otimes v_k\\
&\Psi(\sigma_i)(v_1\dot\otimes\cdots\dot\otimes v_k)=(-1)^{\beta_i\cdot\beta_{i+1}}v_1\dot\otimes\cdots\dot\otimes v_{i+1}\dot\otimes v_i\dot\otimes\cdots\dot\otimes v_k\\
&\qquad\qquad\qquad\qquad(v_i\in V_{\beta_i},\ \beta_i\in\mathbb{Z}_2,\ 1\le i\le k-1)
\end{aligned}$$

where $P=\begin{pmatrix}0 & -\sqrt{-1}I_n\\ \sqrt{-1}I_n & 0\end{pmatrix}\in M(n,n)_1$.

Let W' be a $\mathcal{U}_n$-submodule of W. Since $\mathfrak{q}(n)_0\cong\mathfrak{gl}(n,\mathbb{C})$ as a Lie algebra, and V is a sum of two copies of natural representations of $\mathfrak{gl}(n,\mathbb{C})$ (V_0 and V_1), this embeds $W'|_{\mathfrak{q}(n)_0}$ into a sum of tensor powers of the natural representation, so that this representation of $\mathfrak{gl}(n,\mathbb{C})$ can be integrated to a polynomial representation $\theta_{W'}$ of $GL(n,\mathbb{C})$. Let $\mathrm{Ch}[W']$ denote the character of $\theta_{W'}$, namely $\mathrm{Ch}[W'](x_1,x_2,\ldots,x_n)$ is the trace of $\theta_{W'}(\mathrm{diag}(x_1,x_2,\ldots,x_n))$.

We review Sergeev's result. He established a duality relation between $\mathcal{B}_k$ and $\mathcal{U}_n$ using the double centralizer theorem for semisimple superalgebras, which we will precisely state in the next section, §2.

Theorem 1.10. [11] (1) *Two superalgebras $\mathcal{B}_k$ and $\mathcal{U}_n$ act on W as mutual supercentralizers of each other:*

$$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_n)}(W) = \Psi(\mathcal{B}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}_k)}(W) = \Theta(\mathcal{U}_n).$$

(2) *The simple $\mathcal{B}_k$-module W_ν ($\nu \in DP_k$) occurs in W if and only if $l(\nu) \leq n$.*

(3) *For each $\nu \in DP_k$, let U_ν denote a simple $\mathcal{U}_n$-module corresponding to W_ν in W, in the sense of (the final part of) Theorem 2.1. Then it follows that*

$$\mathrm{Ch}[U_\nu](x_1, x_2, \ldots, x_n) = (\sqrt{2})^{d(\nu)-l(\nu)} Q_\nu(x_1, x_2, \ldots, x_n). \tag{1.10}$$

For each $\nu \in DP_k$ such that $l(\nu)$ is odd, fix $y_\nu \in \mathrm{End}^1_{\mathcal{B}_k}(W_\nu)$ such that $y_\nu^2 = -1$ and let $u_\nu \in \mathrm{End}^1_{\mathcal{U}_n}(U_\nu)$ be defined using y_ν as in (2.2) in which y_i and x_i are replaced by u_ν and y_ν respectively. Then we have

$$W \overset{\cdot}{\cong}_{\mathcal{B}_k \dot{\otimes} \mathcal{U}_n} \bigoplus_{l(\nu):\ \text{even}} W_\nu \dot{\otimes} U_\nu \oplus \bigoplus_{l(\nu):\ \text{odd}} (W_\nu \dot{\otimes} U_\nu)^+ \tag{1.11}$$

where $(W_\nu \dot{\otimes} U_\nu)^\pm$ denotes the simple $\mathcal{B}_k \dot{\otimes} \mathcal{U}_n$-module obtained as the $\pm\sqrt{-1}$-eigenspace of $y_\nu \dot{\otimes} u_\nu \in \mathrm{End}^0_{\mathcal{B}_k \dot{\otimes} \mathcal{U}_n}(W_\nu \dot{\otimes} U_\nu)$ respectively.

2. Double centralizer theorem for semisimple superalgebras

Now we introduce the double supercentralizer theorem for semisimple superalgebras. Although this follows easily from the discussion as in [3], it seems convenient to state it precisely. Here we partly take a finer viewpoint of strict isomorphisms.

Theorem 2.1. *Let A be a semisimple superalgebra. Let V be an A-module with the associated representation $\rho\colon A \to \mathrm{End}(V)$, and*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s \tag{2.1}$$

be the decomposition of V into A-homogeneous components. Put $B = \mathrm{End}^{\cdot}_A(V)$. We define a representation $\tilde{\rho}\colon A \dot{\otimes} B \to \mathrm{End}(V)$ of $A \dot{\otimes} B$ by $\tilde{\rho}(a \dot{\otimes} b) = \rho(a) \circ b$ for all $a \in A$ and $b \in B$. Then each V_i, $1 \leq i \leq s$, is a simple $A \dot{\otimes} B$-submodule of V of type M.

Let U_i be a simple A-module contained in V_i. If U_i is of type M, then there exists a simple B-module W_i of type M such that

$$U_i \dot{\otimes} W_i \overset{\cdot}{\cong}_{A \dot{\otimes} B} \overline{U}_i \dot{\otimes} \overline{W}_i \overset{\cdot}{\cong}_{A \dot{\otimes} B} V_i.$$

If U_i is of type Q, then there exists a simple B-module W_i of type Q such that

$$U_i \dot{\otimes} W_i \overset{\cdot}{\cong}_{A \dot{\otimes} B} V_i \oplus \overline{V}_i.$$

We have

$$U_i \not\cong_A U_j, \quad W_i \not\cong_B W_j$$

for all $1 \le i \ne j \le s$. This gives a $1-1$ correspondence between the isomorphism classes of simple A-modules and simple B-modules appearing in V. Furthermore, we have $\operatorname{End}^{\cdot}_B(V) = \rho(A)$.

Proof. Put $W_i = \operatorname{Hom}^{\cdot}_A(U_i, V_i)$ for each i. Since $b(V_i) \subset V_i$ for any $b \in B$, one can define a representation $\chi_i \colon B \to \operatorname{End}^{\cdot}(W_i)$ by $(\chi_i(b)w)(u) = b(w(u))$ for all $b \in B$, $w \in W_i$, and $u \in U_i$. Then W_i is a simple B-module. In fact, let $W_i' \subset W_i$ be a nonzero B-submodule. Since W_i' is homogeneous, there is a homogeneous $\phi \in W_i' - \{0\}$, which can be regarded as an isomorphism onto $\phi(U_i)$. Let $\phi' \colon \phi(U_i) \to U_i$ be its inverse. Also $V_i = \phi(U_i) \oplus V_i'$ for some A-submodule V_i'. Let ψ be any homogeneous element of W_i. Define $b \in B$ by $b|_{\phi(U_i)} = \psi \circ \phi'$ and $b|_{V_i'} = 0$, whence $\psi = b\phi \in W_i'$. Thus W_i' contains all homogeneous elements of W_i, so that $W_i' = W_i$.

If U_i is of type M (resp. type Q) and $V_i \stackrel{\cdot}{\cong}_A U_i^{\oplus k_i'} \oplus \overline{U}_i^{\oplus l_i'}$ (resp. $V_i \stackrel{\cdot}{\cong}_A U^{\oplus n_i'}$), then by Theorem 1.2 we have $\operatorname{End}^{\cdot}_A(V_i) \cong M(k_i', l_i')$ (resp. $\operatorname{End}^{\cdot}_A(V_i) \cong Q(n_i')$). Since χ_i decomposes as $\operatorname{End}^{\cdot}_A(V) \to \operatorname{End}^{\cdot}_A(V_i) \to \operatorname{End}(W_i)$, the first part being a surjection and $\operatorname{End}^{\cdot}_A(V_i)$ being a simple superalgebra, we have $\chi_i(B) \cong \operatorname{End}^{\cdot}_A(V_i)$. Therefore W_i is of type M (resp. type Q).

Now V_i is a quotient of $U_i \dot{\otimes} W_i$ by an $A \dot{\otimes} B$-homomorphism $u \dot{\otimes} w \to (-1)^{\alpha\beta} w(u)$ for $u \in (U_i)_\alpha$ and $w \in (W_i)_\beta$. If U_i and W_i are of type M, then by Theorem 1.4 $U_i \dot{\otimes} W_i$ is a simple $A \dot{\otimes} B$-module, so that $V_i \stackrel{\cdot}{\cong}_{A\dot{\otimes}B} U_i \dot{\otimes} W_i$. Next suppose U_i and W_i are of type Q. Choose $x_i \in \operatorname{End}^1_A(U_i)$ such that $x_i^2 = -1$, and define $y_i \in \operatorname{End}^1_B(W_i)$ by

$$(y_i(w))(u) = (-1)^\alpha \sqrt{-1}\, w(x_i(u)) \tag{2.2}$$

for all $w \in (W_i)_\alpha$ and $u \in U_i$. Then $y_i^2 = -1$. By Theorem 1.4 and a comparison of dimensions, we have either $V_i \stackrel{\cdot}{\cong}_{A\dot{\otimes}B} (U_i \dot{\otimes} W_i)^+$ or $(U_i \dot{\otimes} W_i)^-$ defined with respect to $x_i \dot{\otimes} y_i$. Direct computation shows that the former is the case.

Consequently we have

$$V \stackrel{\cdot}{\cong}_{A\dot{\otimes}B} \bigoplus_{\text{type } M} U_i \dot{\otimes} W_i \oplus \bigoplus_{\text{type } Q} (U_i \dot{\otimes} W_i)^+. \tag{2.3}$$

Then, using Theorem 1.6 we have

$$\rho(A) \cong \bigoplus_{\text{type } M} M(k_i, l_i) \oplus \bigoplus_{\text{type } Q} Q(n_i)$$

and also by Theorem 1.2

$$B \cong \bigoplus_{\text{type } M} M(k_i', l_i') \oplus \bigoplus_{\text{type } Q} Q(n_i')$$

where

$$\begin{aligned} &\mathbf{dim}\, U_i = (k_i, l_i), \ \ \mathbf{dim}\, W_i = (k_i', l_i') && \text{if } U_i \text{ and } W_i \text{ are of type } M,\\ &\mathbf{dim}\, U_i = (n_i, n_i), \ \ \mathbf{dim}\, W_i = (n_i', n_i') && \text{if } U_i \text{ and } W_i \text{ are of type } Q. \end{aligned}$$

In particular, the B-modules W_i are mutually nonisomorphic. Therefore (2.1) can also be regarded as the decomposition of V into B-homogeneous components. Starting with this and appropriately identifying $A\dot{\otimes}B$-modules with $B\dot{\otimes}A$-modules as noted in §1, **E**, we can see that $\mathrm{End}^{\cdot}_B(V) = \rho(A)$. □

Note that , in (2.3), the signature in $(U_i \dot{\otimes} W_i)^+$ is always $+$ so long as y_i is constructed from x_i by (2.2) regardless of the choice of x_i. The above theorem gives a supercommuting action of two superalgebras A and B on V, which decomposes into a multiplicity-free sum of simple $A\dot{\otimes}B$-modules, where type M (resp. type Q) simple A-modules are paired with type M (resp. type Q) simple B-modules in a bijective manner. Later we also encounter a similar but slightly different situation, in which type M (resp. type Q) simple A-modules are paired with type Q (resp. type M) simple B-modules. Here we formulate this as the following corollary. Note that the superalgebra generated by an element x of degree 1 satisfying $x^2 = -1$ is 2-dimensional, and is isomorphic to $Q(1)$ by $x \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$. It has a unique simple module of dimension 2, which is of type Q. (This superalgebra is also isomorphic to $\mathcal{C}_1$. See the beginning of §3.)

Corollary 2.2. *Let A, V and $V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ be as in Theorem 2.1. Assume that there exists $x \in \mathrm{End}^1_A(V)$ such that $x^2 = -1$. Let C denote the subsuperalgebra of $\mathrm{End}^{\cdot}_A(V)$ generated by x, and put $A' = A \dot{\otimes} C$ and $B = \mathrm{End}^{\cdot}_{A'}(V)$. Restricting the homomorphism $\rho\colon A' \dot{\otimes} B \to \mathrm{End}(V)$ of Theorem 2.1, we can regard V as an $A \dot{\otimes} B$-module. Then each V_i, $1 \le i \le s$, is a simple $A \dot{\otimes} B$-submodule of V of type Q.*

Let T_i be a simple A-module contained in V_i. If T_i is of type M, then there exists a simple B-module W_i of type Q such that

$$T_i \dot{\otimes} W_i \mathrel{\dot{\cong}}_{A\dot{\otimes}B} \overline{T}_i \dot{\otimes} W_i \mathrel{\dot{\cong}}_{A\dot{\otimes}B} V_i.$$

If T_i is of type Q, then there exists a simple B-module W_i of type M such that

$$T_i \dot{\otimes} W_i \mathrel{\dot{\cong}}_{A\dot{\otimes}B} T_i \dot{\otimes} \overline{W}_i \mathrel{\dot{\cong}}_{A\dot{\otimes}B} V_i.$$

We have

$$T_i \not\cong_A T_j, \quad W_i \not\cong_B W_j$$

for all $1 \le i \ne j \le s$. This also gives a $1-1$ correspondence between the isomorphism classes of simple A-modules and simple B-modules appearing in V.

Furthermore, we have $\mathrm{End}^{\cdot}_A(V) = B \dot{\otimes} C$ *and* $\mathrm{End}^{\cdot}_B(V) = \rho(A) \dot{\otimes} C$.

Proof. From Theorem 2.1, V is decomposed as a multiplicity-free sum of simple $A' \dot{\otimes} B$-modules as follows:

$$V \dot{\cong}_{A' \dot{\otimes} B} \bigoplus_{\text{type } M} U_i \dot{\otimes} W_i \oplus \bigoplus_{\text{type } Q} (U_i \dot{\otimes} W_i)^+.$$

Without loss of generality, we may assume that the U_i, $1 \le i \le m$, are of type M and the U_j, $m+1 \le j \le s$, are of type Q for some $m \le s$.

We use Theorem 1.4 for A and C, noting that C has a unique simple module X of dimension 2, which is of type Q. For $1 \le i \le m$, we are in case (c). This implies $k_i = l_i$, and that $T_i = U_i|_A$ is a simple A-module of type Q. Also by Corollary 1.5 the T_i, $1 \le i \le m$, are all mutually nonisomorphic. For $m+1 \le i \le s$, we are in case (b). Hence $U_i \dot{\cong}_{A'} T_i \dot{\otimes} X$ for some simple A-module T_i of type M with dimention (k''_i, l''_i) for some k''_i and l''_i summing up to n_i. We can use $x_i = \mathrm{id}_{V_i} \dot{\otimes} z$, with $z \in \mathrm{End}^1_C(X)$ satisfying $z^2 = -1$, for the x_i required in the proof of Theorem 2.1. Then $(U_i \dot{\otimes} W_i)^+ = T_i \dot{\otimes} (X \dot{\otimes} W_i)^+$. Since $(X \dot{\otimes} W_i)^+|_B \dot{\cong}_B W_i$, we have $(U_i \dot{\otimes} W_i)^+|_{A \dot{\otimes} B} \dot{\cong}_{A \dot{\otimes} B} T_i \dot{\otimes} W_i$. Again by Corollary 1.5, the T_i, $m+1 \le i \le s$, are all mutually nonisomorphic. Therefore we have

$$V \dot{\cong}_{A \dot{\otimes} B} \bigoplus_{1 \le i \le m} T_i \dot{\otimes} W_i \oplus \bigoplus_{m+1 \le i \le s} T_i \dot{\otimes} W_i.$$

in which all T_i are distinct. Thus the $T_i \dot{\otimes} W_i$ are the A-homogeneous components, and this decomposition coincides with that into the V_i in the statement. This establishes a $1-1$ correspondence between the simple A-modules of type Q (resp. type M) and the simple B-modules of type M (resp. type Q) appearing in V.

By Theorem 2.1, we have $\mathrm{End}^{\cdot}_B(V) = \rho(A')$. The surjective homomorphism $\rho(A) \dot{\otimes} C \to \rho(A')$ is an isomorphism, since for any simple component of S of $\rho(A)$, $S \dot{\otimes} C$ is a simple superalgebra by Theorem 1.3, and its image is present since $S \dot{\otimes} 1_C$ is mapped injectively.

Moreover, clearly $\mathrm{End}^{\cdot}_A(V)$ contains C and B, therefore by the same argument $C \dot{\otimes} B$. We have, by Theorem 2.1 and the above decomposition under $A \dot{\otimes} B$,

$$B \cong \bigoplus_{1 \le i \le m} M(k_i, l_i) \oplus \bigoplus_{m+1 \le i \le s} Q(n_i) \quad \text{and}$$

$$\mathrm{End}^{\cdot}_A(V) \cong \bigoplus_{1 \le i \le m} Q(k_i + l_i) \oplus \bigoplus_{m+1 \le i \le s} M(n_i, n_i).$$

Comparing the dimensions, we see that $C \dot{\otimes} B$ gives the whole $\mathrm{End}^{\cdot}_A(V)$. $\square$

Remark. The assumption in the above corollary is actually equivalent to the existence of an invertible element in $\mathrm{End}^1_A(V)$. In fact, the latter means the existence of an invertible element of degree 1 in each simple component of $\mathrm{End}^1_A(V)$, which is equivalent to assuming $k'_i = l'_i$ for all $1 \le i \le m$ in the decomposition of Theorem 2.1 applied for A and $\mathrm{End}^{\cdot}_A(V)$ (not for A' and B). If this is the case, $\mathrm{End}^1_A(V)$ clearly contains an element x' satisfying $(x')^2 = 1$, whence an element x satisfying $x^2 = -1$ also. The other direction is clear since such x is invertible.

3. A simple relation between $\mathcal{A}_k$ and $\mathcal{B}_k$

Let $\mathcal{C}_k$ denote the 2^k-dimensional Clifford algebra, namely $\mathcal{C}_k$ is an associative algebra generated by k elements $\xi_1, \ldots, \xi_k$ subject to relations

$$\xi_i^2 = 1, \quad \xi_i\xi_j = -\xi_j\xi_i \quad (i \neq j) \ . \tag{3.1}$$

We regard $\mathcal{C}_k$ as a superalgebra by giving degree 1 to the generator ξ_i for all $1 \leq i \leq k$. For any subset $I = \{i_1 < i_2 < \cdots < i_r\}$ of $[k] = \{1, 2, \ldots, k\}$, we write $\xi_I = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_r}$. Then $\{\xi_I \ ; \ I \subset [k]\}$ is a basis of $\mathcal{C}_k$. Note that $\mathcal{C}_k$ is isomorphic to a twisted group algebra of $\mathbb{Z}_2^k$ and therefore it is semisimple. Furthermore, it is easy to show that $\mathcal{C}_k$ is a simple superalgebra. If k is even (resp. odd), then $\mathcal{C}_k$ is of type M (resp. of type Q). Put $r = \lfloor k/2 \rfloor$. Define a $\mathbb{Z}_2$-graded simple $\mathcal{C}_k$-module X_k as a minimal left superideal (namely a minimal $\mathbb{Z}_2$-graded left ideal) of $\mathcal{C}_k$ as follows:

$$\begin{aligned} &X_k = \mathcal{C}_k e, \\ &e = e_1e_2\cdots e_r, \quad e_i = \frac{1}{\sqrt{2}}(1 + \sqrt{-1}\xi_{2i-1}\xi_{2i}). \end{aligned} \tag{3.2}$$

For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}_2^r$, put $\xi^\varepsilon = (\xi_1^{\varepsilon_1}e_1)(\xi_3^{\varepsilon_2}e_2)\cdots(\xi_{2r-1}^{\varepsilon_r}e_r)$. If k is even (resp. odd), then $\{\xi^\varepsilon \, ; \, \varepsilon \in \mathbb{Z}_2^r\}$ (resp. $\{\xi^\varepsilon \, ; \, \varepsilon \in \mathbb{Z}_2^r\} \cup \{\xi^\varepsilon\xi_k \, ; \, \varepsilon \in \mathbb{Z}_2^r\}$) is a basis of X_k. If k is odd, define $z_k \in \mathrm{End}^1_{\mathcal{C}_k}(X_k)$ by

$$z_k(\xi^\varepsilon\xi_k^\alpha) = (-1)^{\sum_i \varepsilon_i + \alpha}(\xi^\varepsilon\xi_k^{\alpha+1})$$

for all $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}_2^r$ and $\alpha \in \mathbb{Z}_2$. Note that $z_k^2 = -1$.

Proposition 3.1. (cf. [12, Prop. 3.1]) *The character value of the simple $\mathcal{C}_k$-module X_k is given by*

$$\mathrm{Ch}[X_k]\left(\sum_I c_I\xi_I\right) = 2^{\lfloor (k+1)/2 \rfloor}c_\phi$$

for any element $\sum_I c_I\xi_I \in \mathcal{C}_k$, with $c_I \in \mathbb{C}$ for all $I \subset [k]$.

The following isomorphism is the key to our clarification of the relation between the simple modules over $\mathcal{A}_k$ and $\mathcal{B}_k$.

Theorem 3.2. *There exists a superalgebra homomorphism $\vartheta\colon \mathcal{C}_k \dot{\otimes} \mathcal{A}_k \to \mathcal{B}_k$ satisfying*

$$\begin{aligned} &\vartheta(\xi_1 \dot{\otimes} 1) \mapsto \tau_i \quad (1 \leq i \leq k), \\ &\vartheta(1 \dot{\otimes} \gamma_j) \mapsto \frac{1}{\sqrt{2}}(\tau_j - \tau_{j+1})\sigma_j \quad (1 \leq j \leq k-1) \end{aligned} \tag{3.3}$$

where $\tau_i = \sigma_{i-1}\cdots\sigma_1\tau\sigma_1\ldots\sigma_{i-1}$ for all $1 \leq i \leq k$. Then it follows that ϑ is an isomorphism.

Proof. Put

$$\tilde{\gamma}_j = \frac{1}{\sqrt{2}}(\tau_j - \tau_{j+1})\sigma_j$$

for all $1 \le j \le k-1$.

It is easy to check that the τ_i, $1 \le i \le k$, and the $\tilde{\gamma}_j$, $1 \le j \le k-1$, satisfy the relations (3.1) and (1.2) in which ξ_i and γ_j are replaced by τ_i and $\tilde{\gamma}_j$ respectively. Furthermore, we have $\tau_i \tilde{\gamma}_j = -\tilde{\gamma}_j \tau_i$ for all $1 \le i \le k$, $1 \le j \le k-1$, since

$$\tau_i(\tau_j - \tau_{j+1}) = \begin{cases} -(\tau_j - \tau_{j+1})\tau_{i+1} & \text{if } i = j, \\ -(\tau_j - \tau_{j+1})\tau_{i-1} & \text{if } i = j+1, \\ -(\tau_j - \tau_{j+1})\tau_i & \text{if } i \ne j, j+1, \end{cases}$$

$$\tau_i \sigma_j = \begin{cases} \sigma_j \tau_{i+1} & \text{if } i = j, \\ \sigma_j \tau_{i-1} & \text{if } i = j+1, \\ \sigma_j \tau_i & \text{if } i \ne j, j+1. \end{cases}$$

Therefore ϑ is a homomorphism.

Since

$$\vartheta(\xi_1 \dot{\otimes} 1) = \tau \quad \text{and} \quad \vartheta(\frac{1}{\sqrt{2}}(\xi_j - \xi_{j+1}) \dot{\otimes} \gamma_j) = \sigma_j$$

for all $1 \le j \le k-1$, it follows that ϑ is surjective. Furthermore, it follows that ϑ is bijective, since $\dim(\mathcal{C}_k \dot{\otimes} \mathcal{A}_k) = \dim \mathcal{B}_k = 2^k k!$. Therefore ϑ is an isomorphism. $\square$

Using the above isomorphism, we identify $\mathcal{C}_k \dot{\otimes} \mathcal{A}_k$ with $\mathcal{B}_k$ and regard $\mathcal{C}_k \dot{\otimes} \mathcal{A}_k$-modules $X_k \dot{\otimes} V_\nu$, $\nu \in DP_k$, as $\mathcal{B}_k$-modules. By Theorem 1.4, we can construct all simple $\mathcal{B}_k$-modules as follows. For each $\nu \in DP_k^-$, fix a nonzero element x_ν of $\mathrm{End}^1_{\mathcal{A}_k}(V_\nu)$.

Proposition 3.3. (1) *If k is even, then $\{X_k \dot{\otimes} V_\nu,\ X_k \dot{\otimes} \overline{V}_\nu \mid \nu \in DP_k^+\}$ (resp. $\{X_k \dot{\otimes} V_\nu \mid \nu \in DP_k^-\}$) is a complete set of strict isomorphism classes of simple $\mathcal{B}_k$-modules of type M (resp. type Q).*

(2) *If k is odd, then $\{(X_k \dot{\otimes} V_\nu)^+,\ (X_k \dot{\otimes} V_\nu)^- \mid \nu \in DP_k^-\}$ (resp. $\{X_k \dot{\otimes} V_\nu \mid \nu \in DP_k^+\}$) is a complete set of strict isomorphism clases of simple $\mathcal{B}_k$-modules of type M (resp. type Q), where, if $\nu \in DP_k^-$, $(X_k \dot{\otimes} V_\nu)^\pm$ denotes the $\pm\sqrt{-1}$-eigenspace of $z_k \dot{\otimes} x_\nu \in \mathrm{End}^0_{\mathcal{B}_k}(X_k \dot{\otimes} V_\nu)$ respectively.*

The following proposition shows that our parametrization of the simple $\mathcal{B}_k$-modules coincides with Sergeev's parametrization in Theorem 1.9.

Proposition 3.4. *If k is even or $\nu \in DP_k^+$, then*

$$\mathrm{Ch}[X_k \dot{\otimes} V_\nu] = \psi_\nu.$$

If k is odd and $\nu \in DP_k^-$, then

$$\mathrm{Ch}[(X_k \dot{\otimes} V_\nu)^+] = \mathrm{Ch}[(X_k \dot{\otimes} V_\nu)^-] = \psi_\nu.$$

Here ψ_ν is the irreducible character defined in Theorem 1.9.

Proof. Let $\nu \in DP_k$, $\mu \in OP_k$ and put $l = l(\mu)$. Since $\sigma^{(\mu,\phi)}$ is a product of $k-l$ generators σ_j, its image under ϑ^{-1} is a product of $k-l$ elements of $\mathcal{C}_k \dot{\otimes} \mathcal{A}_k$ of the form $\frac{1}{\sqrt{2}}(\xi_j - \xi_{j+1}) \dot{\otimes} \gamma_j$. Rearrange this product into the form

$$(\text{constant}) \times (\text{product of the } \xi_j - \xi_{j+1}) \dot{\otimes} (\text{product of the } \gamma_j).$$

Let Ξ denote the product of the $\xi_j - \xi_{j+1}$ in this expression. The product of the γ_j equals γ^μ. Therefore we have

$$\vartheta^{-1}(\sigma^{(\mu,\phi)}) = (-1)^{\binom{k-l}{2}} \left(\frac{1}{\sqrt{2}}\right)^{k-l} \Xi \mathbin{\dot{\otimes}} \gamma^\mu,$$

where the signature comes from interchanging the elements of degree 1. This gives

$$\mathrm{Ch}[X_k \mathbin{\dot{\otimes}} V_\nu](\vartheta^{-1}(\sigma^{(\mu,\phi)}) = (-1)^{\binom{k-l}{2}} \left(\frac{1}{\sqrt{2}}\right)^{k-l} \mathrm{Ch}[X_k](\Xi)\,\mathrm{Ch}[V_\nu](\gamma^\mu).$$

By Proposition 3.1, $\mathrm{Ch}[X_k](\Xi)$ equals $2^{\lfloor (k+1)/2 \rfloor}$ times the coefficient of 1 in the expansion of this product into the ξ_I, $I \subset [k]$. We have

$$\Xi = \alpha_1(\Xi_1)\alpha_2(\Xi_2)\cdots\alpha_l(\Xi_l)$$

where

$$\Xi_j = (\xi_1 - \xi_2)(\xi_2 - \xi_3)\cdots(\xi_{\mu_j - 1} - \xi_{\mu_j}) \in \mathcal{C}_{\mu_j}$$

and $\alpha_j \colon \mathcal{C}_{\mu_j} \to \mathcal{C}_k$ is the embedding defined by $\xi_i \mapsto \xi_{\mu_1+\mu_2+\cdots+\mu_{j-1}+i}$ $(1 \leq i \leq \mu_j, 1 \leq j \leq l)$. If we let c_j denote the coefficient of 1 in the expansion of Ξ_j into the ξ_{I_j}, $I_j \subset [\mu_j]$, then we have $\mathrm{Ch}[X_k](\Xi) = 2^{\lfloor (k+1)/2 \rfloor} c_1 c_2 \cdots c_l$. Clearly

$$c_j = \begin{cases} 0 & \text{if } \mu_j \text{ is even,} \\ (-1)^{\frac{\mu_j - 1}{2}} & \text{if } \mu_j \text{ is odd,} \end{cases}$$

so that we have $\mathrm{Ch}[X_k](\Xi) = 2^{\lfloor (k+1)/2 \rfloor}(-1)^{(k-l)/2}$. This signature cancels with $(-1)^{\binom{k-l}{2}}$, since $\binom{k-l}{2} + \frac{k-l}{2} = \frac{(k-l)^2}{2}$ and we have $k - l \equiv 0 \pmod 2$ (so that $(k-l)^2 \equiv 0 \pmod 4$) because $\mu \in OP_k$. Therefore we have

$$\begin{aligned} &\mathrm{Ch}[X_k \mathbin{\dot{\otimes}} V_\nu](\vartheta^{-1}(\sigma^{(\mu,\phi)})) \\ &= 2^{\lfloor \frac{k+1}{2} \rfloor}(\sqrt{2})^{l-k}\,\mathrm{Ch}[V_\nu](\gamma^\mu) = \begin{cases} (\sqrt{2})^{l}\,\mathrm{Ch}[V_\nu](\gamma^\mu) & \text{if } k \text{ is even,} \\ \\ (\sqrt{2})^{l+1}\,\mathrm{Ch}[V_\nu](\gamma^\mu) & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

In particular, if k is odd and $\nu \in DP_k^-$, we have

$$\begin{aligned} &\mathrm{Ch}[(X_k \mathbin{\dot{\otimes}} V_\nu)^+](\vartheta^{-1}(\sigma^{(\mu,\phi)})) = \mathrm{Ch}[(X_k \mathbin{\dot{\otimes}} V_\nu)^-](\vartheta^{-1}(\sigma^{(\mu,\phi)})) \\ &= (\sqrt{2})^{l-1}\,\mathrm{Ch}[V_\nu](\gamma^\mu). \end{aligned}$$

If k is even, then $d(\nu) = \varepsilon(\nu)$ for any $\nu \in DP_k$. By Theorem 1.8, Theorem 1.9 and Theorem 1.10, we have

$$\begin{aligned} &\sum_{\nu \in DP_k} \mathrm{Ch}[X_k \mathbin{\dot{\otimes}} V_\nu](\vartheta^{-1}(\sigma^{(\mu,\phi)}))(\sqrt{2})^{-l(\nu)-d(\nu)} Q_\nu \\ &= (\sqrt{2})^{l(\mu)} \sum_{\nu \in DP_k} \mathrm{Ch}[V_\nu](\gamma^\mu)(\sqrt{2})^{-l(\nu)-\varepsilon(\nu)} Q_\nu \\ &= 2^{l(\mu)} p_\mu \\ &= \sum_{\nu \in DP_k} \psi_\nu(\sigma^{(\mu,\phi)})(\sqrt{2})^{-l(\nu)-d(\nu)} Q_\nu. \end{aligned}$$

Since the Q_ν, $\nu \in DP_k$, are linearly independent in Ω^k (cf. §1, **C**), we have $\mathrm{Ch}[X_k \dot{\otimes} V_\nu](\vartheta^{-1}(\sigma^{(\mu,\phi)})) = \psi_\nu(\sigma^{(\mu,\phi)})$. Therefore we have $\mathrm{Ch}[X_k \dot{\otimes} V_\nu] = \psi_\nu$.

If k is odd, then $d(\nu) + \varepsilon(\nu) = 1$ for any $\nu \in DP_k$. We have

$$\begin{aligned}
&\sum_{\nu \in DP_k} 2^{-\varepsilon(\nu)} \mathrm{Ch}[X_k \dot{\otimes} V_\nu](\vartheta^{-1}(\sigma^{(\mu,\phi)}))(\sqrt{2})^{-l(\nu)-d(\nu)} Q_\nu \\
&= \sum_{\nu \in DP_k} (\sqrt{2})^{l+1-2\varepsilon(\nu)} \mathrm{Ch}[V_\nu](\gamma^\mu)(\sqrt{2})^{-l(\nu)-d(\nu)} Q_\nu \\
&= 2^{l(\mu)} p_\mu \\
&= \sum_{\nu \in DP_k} \psi_\nu(\sigma^{(\mu,\phi)})(\sqrt{2})^{-l(\nu)-d(\nu)} Q_\nu.
\end{aligned}$$

This gives $2^{-\varepsilon(\nu)} \mathrm{Ch}[X_k \dot{\otimes} V_\nu] = \psi_\nu$. Namely, $\mathrm{Ch}[X_k \dot{\otimes} V_\nu] = \psi_\nu$ if $\nu \in DP_k^+$ and $\mathrm{Ch}[(X_k \dot{\otimes} V_\nu)^+] = \mathrm{Ch}[(X_k \dot{\otimes} V_\nu)^-] = \psi_\nu$ if $\nu \in DP_k^-$. □

By Proposition 3.4 , we can fix the choice of simple $\mathcal{B}_k$-modules W_ν, which afford the characters ψ_ν, as follows:

$$W_\nu = \begin{cases} X_k \dot{\otimes} V_\nu & \text{if } k \text{ is even or } \nu \in DP_k^+, \\ (X_k \dot{\otimes} V_\nu)^+ & \text{if } k \text{ is odd and } \nu \in DP_k^-, \end{cases} \tag{3.4}$$

where $(X_k \dot{\otimes} V_\nu)^\pm$ are as in Proposition 3.3. The W_ν, $\nu \in DP_k$, form a complete set of (not strict) isomorphism classes of simple $\mathcal{B}_k$-module.

The following proposition gives the restriction rule of the simple $\mathcal{B}_k$-modules to $\mathcal{A}_k$ by $\mathcal{A}_k \xrightarrow{\cong} 1 \dot{\otimes} \mathcal{A}_k \subset \mathcal{C}_k \dot{\otimes} \mathcal{A}_k \xrightarrow{\vartheta} \mathcal{B}_k$.

Proposition 3.5. *For any $\nu \in DP_k$, the simple $\mathcal{B}_k$-module W_ν restricts to an $\mathcal{A}_k$-module as follows:*

$$W_\nu|_{\mathcal{A}_k} \cong \overline{W_\nu|_{\mathcal{A}_k}} \cong \begin{cases} (V_\nu \oplus \overline{V}_\nu)^{\oplus 2^{\lfloor (k-1)/2 \rfloor}} & \text{if } \nu \in DP_k^+, \\ V_\nu^{\oplus 2^{\lfloor (k+1)/2 \rfloor}} & \text{if } \nu \in DP_k^-,\ k \text{ is even}, \\ V_\nu^{\oplus 2^{\lfloor (k-1)/2 \rfloor}} & \text{if } \nu \in DP_k^-,\ k \text{ is odd}, \end{cases}$$

where $\overline{W_\nu|_{\mathcal{A}_k}}$ and $\overline{V}_\nu$ are shifts of $W_\nu|_{\mathcal{A}_k}$ and V_ν respectively.

Proof. First, in view of Theorem 1.8, the $\mathcal{A}_k$-modules on the right-hand sides are all strictly isomorphic to their shifts. Assume $\nu \in DP_k^+$. We have $\xi \dot{\otimes} V_\nu \overset{.}{\cong} V_\nu$ (resp. $\overline{V}_\nu$) if $\xi \in (X_k)_0$ (resp. $(X_k)_1$). From (3.4) and the fact that $\dim(X_k)_0 = \dim(X_k)_1 = 2^{\lfloor (k-1)/2 \rfloor}$, we have

$$W_\nu|_{\mathcal{A}_k} \overset{.}{\cong} V_\nu^{\oplus 2^{\lfloor (k-1)/2 \rfloor}} \oplus \overline{V}_\nu^{\oplus 2^{\lfloor (k-1)/2 \rfloor}}.$$

Assume $\nu \in DP_k^-$. We have $\xi \dot{\otimes} V_\nu \overset{.}{\cong} V_\nu \overset{.}{\cong} \overline{V}_\nu$ for any homogeneous element $\xi \in X_k$. Therefore we have $W_\nu|_{\mathcal{A}_k} = V_\nu^{\oplus 2^{\lfloor (k+1)/2 \rfloor}}$ if k is even, and we have $W_\nu|_{\mathcal{A}_k} \oplus \overline{W}_\nu|_{\mathcal{A}_k} = X_k \dot{\otimes} V_\nu|_{\mathcal{A}_k} = V_\nu^{\oplus 2^{\lfloor (k+1)/2 \rfloor}}$ if k is odd. Note that $\overline{W}_\nu = (X_k \dot{\otimes} V_\nu)^-$. The result also follows in this case. □

4. A duality relation of $\mathcal{A}_k$ and $\mathfrak{q}(n)$

In this section, we establish a duality relation between $\mathcal{A}_k$ and $\mathcal{U}_n$.

We can rewrite (1.11) using the description of W_ν after the Proof of Proposition 3.4. If k is even, then we have

$$W \dot{\cong} \bigoplus_{\nu \in DP_k^+} (X_k \dot{\otimes} V_\nu) \dot{\otimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} \Big((X_k \dot{\otimes} V_\nu) \dot{\otimes} U_\nu\Big)^+$$

and if k is odd, then we have

$$W \dot{\cong} \bigoplus_{\nu \in DP_k^-} (X_k \dot{\otimes} V_\nu)^+ \dot{\otimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^+} \Big((X_k \dot{\otimes} V_\nu) \dot{\otimes} U_\nu\Big)^+ .$$

Note that the symbol $^+$ is used in three ways.

(1) If k is odd and $\nu \in DP_k^-$, then both X_k and V_ν are of type Q. $(X_k \dot{\otimes} V_\nu)^+$ is the $(+\sqrt{-1})$-eigenspace of $z_k \dot{\otimes} x_\nu$.

(2) If k is even and $\nu \in DP_k^-$, then X_k, V_ν, U_ν are of type M, Q, Q respectively. $((X_k \dot{\otimes} V_\nu) \dot{\otimes} U_\nu)^+$ is the $(+\sqrt{-1})$-eigenspace of $(1 \dot{\otimes} x_\nu) \dot{\otimes} u_\nu$, where $u_\nu \in \mathrm{End}^1_{\mathcal{U}_n}(U_\nu)$ is defined from $1 \dot{\otimes} x_\nu$ as in (2.2).

(3) If k is odd and $\nu \in DP_k^+$, then X_k, V_ν, U_ν are of type Q, M, Q respectively. $((X_k \dot{\otimes} V_\nu) \dot{\otimes} U_\nu)^+$ is the $(+\sqrt{-1})$-eigenspace of $(z_k \dot{\otimes} 1) \dot{\otimes} u_\nu$, where $u_\nu \in \mathrm{End}^1_{\mathcal{U}_n}(U_\nu)$ is defined from $z_k \dot{\otimes} 1$ as in (2.2).

Put $r = \lfloor k/2 \rfloor$ and $\zeta_i = \sqrt{-1}\xi_{2i-1}\xi_{2i} \in \mathcal{C}_k$ for $1 \le i \le r$. The $\Psi(\zeta_i)$, $1 \le i \le r$, are commuting involutions of $\Psi((\mathcal{C}_k)_0) \subset \Psi((\mathcal{B}_k)_0) = \mathrm{End}^0_{\Theta(\mathcal{U}_n)}(W)$.

Then W is a direct sum of the simultaneous eigenspaces W^ε, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}_2^r$, namely we have

$$W = \bigoplus_{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}_2^r} W^\varepsilon,$$
$$W^\varepsilon = \{ w \in W \,;\, \Psi(\zeta_i)(w) = (-1)^{\varepsilon_i} w \quad (1 \le i \le r) \}.$$

It is clear that W^ε is an $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-module for each $\varepsilon \in \mathbb{Z}_2^r$.

Theorem 4.1. *For each $\varepsilon \in \mathbb{Z}_2^r$, the submodule W^ε is decomposed as a multiplicity-free sum of simple $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-modules, in which the simple $\mathcal{A}_k$-modules are paired with the simple $\mathcal{U}_n$-modules in a bijective manner ((4.1) and (4.3)). More precisely, we have the following decomposition.*

For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}_2^r$, put $\alpha(\varepsilon) = \varepsilon_1 + \cdots + \varepsilon_r \in \mathbb{Z}_2$.

(1) *Assume that k is even. If $\alpha(\varepsilon) = 0$, then W^ε is a direct sum of simple $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-modules of type M as follows:*

$$W^\varepsilon \dot{\cong}_{\mathcal{A}_k \dot{\otimes} \mathcal{U}_n} \bigoplus_{\nu \in DP_k^+} V_\nu \dot{\otimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} (V_\nu \dot{\otimes} U_\nu)^+ \tag{4.1}$$

where $(V_\nu \dot{\otimes} U_\nu)^\pm$ *denotes the* $\pm\sqrt{-1}$*-eigenspace of* $x_\nu \dot{\otimes} u_\nu \in \mathrm{End}^0_{\mathcal{A}_k \dot{\otimes} \mathcal{U}_n}(V_\nu \dot{\otimes} U_\nu)$ *, where* u_ν *is defined as in Theorem 1.10 with* $y_\nu = 1 \dot{\otimes} x_\nu$ *(if* k *is even and* $\nu \in DP_k^-$*) or* $y_\nu = z_k \dot{\otimes} 1$ *(if* k *is odd and* $\nu \in DP_k^+$*).*

If $\alpha(\varepsilon) = 1$*, then* W^ε *is also a direct sum of simple* $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$*-modules of type* M *and is strictly isomorphic to the shift of the module on the right-hand side in* (4.1)*.*

Furthermore we have

$$\mathrm{End}^{\cdot}_{\Theta(U_n)}(W^\varepsilon) = \Psi(\mathcal{A}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) = \Theta(\mathcal{U}_n) \tag{4.2}$$

for all $\varepsilon \in \mathbb{Z}_2^r$*.*

(2) *Assume that* k *is odd. Then* W^ε *is an* $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$*-module of type* Q *and*

$$W^\varepsilon \cong_{\mathcal{A}_k \dot{\otimes} \mathcal{U}_n} \bigoplus_{\nu \in DP_k} V_\nu \dot{\otimes} U_\nu. \tag{4.3}$$

Furthermore we have

$$\mathrm{End}^{\cdot}_{\Theta(U_n)}(W^\varepsilon) \cong \mathcal{C}_1 \otimes \Psi(\mathcal{A}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) \cong \mathcal{C}_1 \otimes \Theta(\mathcal{U}_n). \tag{4.4}$$

Proof. For each $\varepsilon \in \mathbb{Z}_2^r$, let X_k^ε denote the simultaneous eigenspace of X_k of the $\zeta_i = \sqrt{-1}\xi_{2i-1}\xi_{2i}$, $1 \le i \le r$, namely

$$X_k^\varepsilon = \{\xi \in X_k \,;\, \zeta_i \xi = (-1)^{\varepsilon_i} \xi \quad (1 \le i \le r)\}.$$

(1) Assume that k is even. Then we have $X_k^\varepsilon = \mathbb{C}\xi^\varepsilon$ since $\zeta_i \xi^\varepsilon = (-1)^{\varepsilon_i} \xi^\varepsilon$ for each $\varepsilon \in \mathbb{Z}_2^r$.

If $\nu \in DP_k^-$, then we have

$$(X_k \dot{\otimes} V_\nu \dot{\otimes} U_\nu)^+ = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} X_k^\varepsilon \dot{\otimes} (V_\nu \dot{\otimes} U_\nu)^+$$

since the ζ_i, $1 \le i \le r$, and $1 \dot{\otimes} x_\nu \dot{\otimes} u_\nu$ commute. If $\nu \in DP_k^+$, then we have

$$X_k \dot{\otimes} V_\nu \dot{\otimes} U_\nu = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} X_k^\varepsilon \dot{\otimes} (V_\nu \dot{\otimes} U_\nu).$$

Consequently,

$$\begin{aligned} W^\varepsilon &\dot{\cong}_{\mathcal{A}_k \dot{\otimes} \mathcal{U}_n} \xi^\varepsilon \dot{\otimes} \left(\bigoplus_{\nu \in DP_k^+} V_\nu \dot{\otimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} (V_\nu \dot{\otimes} U_\nu)^+ \right) \\ &\dot{\cong}_{\mathcal{A}_k \dot{\otimes} \mathcal{U}_n} \begin{cases} \bigoplus_{\nu \in DP_k^+} V_\nu \dot{\otimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} (V_\nu \dot{\otimes} U_\nu)^+ & \text{if } \alpha(\varepsilon) = 0, \\ \overline{\bigoplus_{\nu \in DP_k^+} V_\nu \dot{\otimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} (V_\nu \dot{\otimes} U_\nu)^+} & \text{if } \alpha(\varepsilon) = 1. \end{cases} \end{aligned}$$

Therefore (4.1) follows.

Since W^ε is an $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-module, we have

$$\Theta(\mathcal{U}_n)|_{W^\varepsilon} \subset \operatorname{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon).$$

Define a linear map $\mathfrak{p}_\varepsilon \colon \Theta(\mathcal{U}_n) \to \Theta(\mathcal{U}_n)|_{W^\varepsilon}$ by $\mathfrak{p}_\varepsilon(f) = f|_{W^\varepsilon}$ for all $f \in \Theta(\mathcal{U}_n)$. We claim that $\mathfrak{p}_\varepsilon$ is injective. Assume that $f \in \ker \mathfrak{p}_\varepsilon$, namely $\mathfrak{p}_\varepsilon(f) = 0 \in \operatorname{End}(W^\varepsilon)$. Since f and ξ_{2j-1}'s commute, and a subgroup of $(\mathcal{C}_k)^\times$ generated by the ξ_{2j-1}, $1 \le j \le r$, transitively act on $\{W^{\varepsilon'}\,;\, \varepsilon' \in \mathbb{Z}_2^r\}$ as follows:

$$\xi_{2j-1} W^{(\varepsilon_1,\ldots,\varepsilon_r)} = W^{(\varepsilon_1,\ldots,\varepsilon_j+1,\ldots,\varepsilon_r)}$$

for all $1 \le j \le r$, it follows that $f|_{W^{\varepsilon'}} = 0$ for all $\varepsilon' \in \mathbb{Z}_2^r$. Therefore $f = 0$ in $\operatorname{End}(W)$, as required. Hence $\mathfrak{p}_\varepsilon$ is injective.

From Theorem 1.10, Theorem 2.1, and (4.1), we have

$$\dim \operatorname{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) = \dim \operatorname{End}^{\cdot}_{\Psi(\mathcal{B}_k)}(W) = \dim \Theta(\mathcal{U}_n) = \dim \Theta(\mathcal{U}_n)|_{W^\varepsilon}.$$

Consequently we have

$$\operatorname{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) = \Theta(\mathcal{U}_n)|_{W^\varepsilon}$$

and, from Theorem 2.1, we have

$$\operatorname{End}^{\cdot}_{\Theta(\mathcal{U}_n)}(W^\varepsilon) = \Psi(\mathcal{A}_k).$$

(2) Assume that k is odd. If $\nu \in DP_k^+$, then we regard the $\mathcal{A}_k \dot{\otimes} \mathcal{C}_k$-module $V_\nu \dot{\otimes} X_k$ as a $\mathcal{C}_k \dot{\otimes} \mathcal{A}_k$-module via $\omega_{\mathcal{C}_k,\mathcal{A}_k}$ (cf. §1, **E**). Then $X_k \dot{\otimes} V_\nu \dot{\cong} V_\nu \dot{\otimes} X_k$ by the map $\theta \colon X_k \dot{\otimes} V_\nu \to V_\nu \dot{\otimes} X_k$ defined by $\theta(\xi \dot{\otimes} v) = (-1)^{\alpha\cdot\beta} v \dot{\otimes} \xi$ for all homogeneous $\xi \in (X_k)_\alpha$ and $v \in (V_\nu)_\beta$ (α, $\beta \in \mathbb{Z}_2$). Since $\theta \circ (z_k \dot{\otimes} 1) = (1 \dot{\otimes} z_k) \circ \theta$, we have

$$(X_k \dot{\otimes} V_\nu \dot{\otimes} U_\nu)^+ \dot{\cong}_{\mathcal{B}_k \dot{\otimes} \mathcal{U}_n} V_\nu \dot{\otimes} (X_k \dot{\otimes} U_\nu)^+$$

where $(X_k \dot{\otimes} U_\nu)^\pm$ denotes the $\pm\sqrt{-1}$-eigenspace of $z_k \dot{\otimes} u_\nu \in \operatorname{End}^0_{\mathcal{C}_k \dot{\otimes} \mathcal{U}_n}(X_k \dot{\otimes} U_\nu)$ respectively. Since the ζ_i, $1 \le i \le r$, and $z_k \dot{\otimes} u_\nu$ commute, we have

$$(X_k \dot{\otimes} U_\nu)^+ = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} (X_k^\varepsilon \dot{\otimes} U_\nu)^+$$

where $(X_k^\varepsilon \dot{\otimes} U_\nu)^\pm$ denotes the $\pm\sqrt{-1}$-eigenspace of $z_k|_{X_k^\varepsilon} \dot{\otimes} u_\nu \in \operatorname{End}^0_{\mathcal{U}_n}(X_k^\varepsilon \dot{\otimes} U_\nu)$ respectively. Note that we have $X_k^\varepsilon = \mathbb{C}\xi^\varepsilon \oplus \mathbb{C}\xi^\varepsilon \xi_k$, since $\zeta_i(\xi^\varepsilon \xi_k^\alpha) = (-1)^{\varepsilon_i} \xi^\varepsilon \xi_k^\alpha$ for each $\varepsilon \in \mathbb{Z}_2^r$ and $\alpha \in \mathbb{Z}_2$. Then it is clear that $(X_k^\varepsilon \dot{\otimes} U_\nu)^+$ is a $\mathcal{U}_n$-module for each $\varepsilon \in \mathbb{Z}_2^r$. Moreover, by theorem 1.4 (c) we have

$$(X_k \dot{\otimes} U_\nu)^+ \dot{\cong}_{\mathcal{U}_n} U_\nu^{\oplus 2^r}.$$

Therefore we have

$$(X_k^\varepsilon \dot{\otimes} U_\nu)^+ \dot{\cong}_{\mathcal{U}_n} U_\nu$$

for all $\varepsilon \in \mathbb{Z}_2^r$. If $\nu \in DP_k^-$, then we have

$$(X_k \dot{\otimes} V_\nu)^+ \dot{\otimes} U_\nu = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} (X_k^\varepsilon \dot{\otimes} V_\nu)^+ \dot{\otimes} U_\nu$$

since the ζ_i, $1 \leq i \leq r$, and $(z_k \dot{\otimes} x_\nu) \dot{\otimes} 1$ commute, where $(X_k^\varepsilon \dot{\otimes} V_\nu)^\pm$ denotes the $\pm\sqrt{-1}$-eigenspace of $z_k|_{X_k^\varepsilon} \dot{\otimes} x_\nu \in \mathrm{End}^0_{\mathcal{A}_k}(X_k^\varepsilon \dot{\otimes} V_\nu)$ respectively. By Theorem 1.4 (c) we have

$$(X_k^\varepsilon \dot{\otimes} V_\nu)^+ \dot{\cong}_{\mathcal{A}_k} V_\nu.$$

Consequently we have

$$W^\varepsilon \dot{\cong} \overline{W^\varepsilon} \dot{\cong} \bigoplus_{\nu \in DP_k} V_\nu \dot{\otimes} U_\nu.$$

Therefore (4.3) follows. From Corollary 2.2, (4,4) follows. $\square$

Corollary 4.2. *The duality relation of $\mathcal{A}_k$ and $\mathcal{U}_n$ in Theorem 4.1 gives Schur's formula* (1.6)*. Namely we get Schur's formula* (1.6) *by the calculation of the character of the $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-module W^ε in Theorem 4.1.*

Proof. By what we noted before Theorem 1.10, any $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-submodule W' of W can be regarded as an $\mathcal{A}_k$-module with a commuting polynomial representation $\theta_{W'}$ of $GL(n, \mathbb{C})$. Here we extend our notation in Theorem 1.10 to let $\mathrm{Ch}[W'] \in Z_0((\mathcal{A}_k \otimes \mathbb{C}[GL(n,\mathbb{C})]^*)$ be determined by $x \otimes g \mapsto \mathrm{tr}(x_{W'} \circ \theta_{W'}(g))$ for $x \in \mathcal{A}_k$ and $g \in GL(n, \mathbb{C})$, where $x_{W'}$ denotes the action of $x \in \mathcal{A}_k$ on W'.

For any ε, $\varepsilon' \in \mathbb{Z}_2^r$, we have $\mathrm{Ch}[W^\varepsilon] = \mathrm{Ch}[W^{\varepsilon'}]$, since $W^\varepsilon \cong W^{\varepsilon'}$. Then, we have $\mathrm{Ch}[W] = 2^r \,\mathrm{Ch}[W^\varepsilon]$ for any $\varepsilon \in \mathbb{Z}_2^r$. Therefore, for each $\mu \in OP_k$ and each diagonal element $E = \mathrm{diag}\,(x_1, x_2, \ldots, x_n) \in GL(n, \mathbb{C})$, we have

$$\mathrm{Ch}[W^\varepsilon]\,(\gamma^\mu \otimes E) = 2^{-r}\,\mathrm{Ch}[W]\left(\vartheta(1 \dot{\otimes} \gamma^\mu) \otimes E\right).$$

Put $l = l(\mu)$. Since $1 \dot{\otimes} \gamma^\mu$ is a product of $k - l$ elements $1 \dot{\otimes} \gamma_j$, its image under ϑ is a product of $k - l$ elements of $\mathcal{B}_k$ of the form $\frac{1}{\sqrt{2}}(\tau_j - \tau_{j+1})\sigma_j$. Rearrange this product into the form

$$(\text{constant}) \times (\text{product of the } \tau_r - \tau_s) \times (\text{product of the } \sigma_j).$$

The product of the σ_j equals $\sigma^{(\mu,\phi)}$. Expanding the product of $\tau_r - \tau_s$ into a sum of $k - l$ elements, we have

$$\vartheta(1 \dot{\otimes} \gamma^\mu) = \left(\frac{1}{\sqrt{2}}\right)^{k-l} \times \sum (\text{product of the } \tau_r) \times \sigma^{(\mu,\phi)}.$$

Then all terms in the summation are conjugate to $\sigma^{(\mu,\phi)}$ in $(\mathcal{B}_k)^\times$. Therefore we have

$$\begin{aligned}\mathrm{Ch}[W^\varepsilon]\,(\gamma^\mu \otimes E) &= 2^{-r}2^{k-l}(\sqrt{2})^{l-k}\,\mathrm{Ch}[W]\left(\sigma^{(\mu,\phi)} \otimes E\right) \\ &= (\sqrt{2})^{-2r+k-l)}2^l p_\mu(x_1,\dots,x_n) \\ &= \begin{cases} (\sqrt{2})^l p_\mu(x_1,\dots,x_n) & \text{if } k \text{ is even,} \\ (\sqrt{2})^{1+l} p_\mu(x_1,\dots,x_n) & \text{if } k \text{ is odd.}\end{cases}\end{aligned}$$

On the other hand, using (1.10), if k is even, then we have

$$\begin{aligned}&\mathrm{Ch}[\bigoplus_{\nu\in DP_k^+} V_\nu \dot{\otimes} U_\nu \oplus \bigoplus_{\nu\in DP_k^-} (V_\nu \dot{\otimes} U_\nu)^+]\,(\gamma^\mu \otimes E) \\ &= \sum_{\nu\in DP_k} \mathrm{Ch}[V_\nu](\gamma^\mu)(\sqrt{2})^{-d(\nu)-l(\nu)}Q_\nu(x_1,\dots,x_n) \\ &= \sum_{\nu\in DP_k} \mathrm{Ch}[V_\nu](\gamma^\mu)(\sqrt{2})^{-\varepsilon(\nu)-l(\nu)}Q_\nu(x_1,\dots,x_n)\end{aligned}$$

and if k is odd, then we have

$$\begin{aligned}&\mathrm{Ch}[\bigoplus_{\nu\in DP_k^+} V_\nu \dot{\otimes} U_\nu \oplus \bigoplus_{\nu\in DP_k^-} V_\nu \dot{\otimes} U_\nu]\,(\gamma^\mu \otimes E) \\ &= \sum_{\nu\in DP_k^+} \mathrm{Ch}[V_\nu](\gamma^\mu)(\sqrt{2})^{1-l(\nu)}Q_\nu(x_1,\dots,x_n) \\ &\qquad + \sum_{\nu\in DP_k^-} \mathrm{Ch}[V_\nu](\gamma^\mu)(\sqrt{2})^{-l(\nu)}Q_\nu(x_1,\dots,x_n) \\ &= \sqrt{2} \times \sum_{\nu\in DP_k} \mathrm{Ch}[V_\nu](\gamma^\mu)(\sqrt{2})^{-\varepsilon(\nu)-l(\nu)}Q_\nu(x_1,\dots,x_n).\end{aligned}$$

Consequently the result follows. □

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